Matrix operations & manipulating matrices

Beezer: p125 – 147 print version
Strang: Sec 1.4 & 1.6.

Definition: M_{mn} is the set of all $m \times n$ matrices.

Matrix Equality "what does it mean to say that two matrices are equal?"

Definition: We say $A = B$ if $a_{11} = b_{11}$
$a_{12} = b_{12}$
$\vdots$
$a_{ij} = b_{ij}$ for all i, j.

$$\begin{bmatrix} a_{11} & \cdots & a_{1n} \\ & \vdots & \\ \end{bmatrix} \quad \begin{bmatrix} b_{11} & \cdots & b_{1n} \\ & & \\ \end{bmatrix}$$

$\begin{bmatrix} a_{m1} & \cdots & a_{mn} \end{bmatrix} \quad \begin{bmatrix} b_{m1} & & b_{mn} \end{bmatrix}$

i.e. $\boxed{A = B \text{ if } [A]_{ij} = [B]_{ij} \text{ for all } i,j}$

Ex: $A = \begin{bmatrix} 3 & 2 \\ 1 & 1 \end{bmatrix} \qquad B = \begin{bmatrix} 3 & 2 \\ 1 & 2 \end{bmatrix}$

$A \neq B$ because $[A]_{22} = 1 \neq 2 = [B]_{22}$

Definition: Matrix addition: suppose A, B are $m \times n$ matrices.

$A + B$ is the $m \times n$ matrix with entries

$[A+B]_{ij} = [A]_{ij} + [B]_{ij}$.

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Ex: $A = \begin{bmatrix} 1 & 1 \\ 2 & 0 \end{bmatrix}$ $B = \begin{bmatrix} 0 & 1 \\ 1 & 3 \end{bmatrix}$

$$A+B = \begin{bmatrix} 1+0 & 1+1 \\ 2+1 & 0+3 \end{bmatrix} = \begin{bmatrix} 1 & 2 \\ 3 & 3 \end{bmatrix}$$

Note: we can only add matrices of the same size!

Ex: If $A = \begin{bmatrix} 2 & 2 \\ 3 & 1 \\ 2 & 1 \end{bmatrix}$ $B = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$,

$A+B$ is not defined.

Definition: Scalar multiplication.

Definition:

Let α be a scalar (i.e. a real number). We write this as $\alpha \in \mathbb{R}$. Let $A \in M_{mn}$ be an $m \times n$ matrix.

Then αA is <u>also</u> an $m \times n$ matrix with

$$[\alpha A]_{ij} = \alpha [A]_{ij}.$$

Example: $\alpha = 2$ $\quad A = \begin{bmatrix} 3 & 3 & 1 \\ 2 & 0 & 1 \end{bmatrix}$

$$\alpha A = \begin{bmatrix} 2\times 3 & 2\times 3 & 2\times 1 \\ 2\times 2 & 2\times 0 & 2\times 1 \end{bmatrix} = \begin{bmatrix} 6 & 6 & 2 \\ 4 & 0 & 2 \end{bmatrix}$$

Combine addition & scalar multiplication:

Let A, B be $m \times n$ matrices & let $\alpha \in \mathbb{R}$.

$A + \underline{\alpha B}$: We'll get back to this shortly.

Recall that there is a real number (0) that satisfies $0 + r = r$ for all $r \in \mathbb{R}$.

Definition: the $m \times n$ zero matrix $\mathcal{O}$ or $\mathcal{O}_{m\times n}$ is the matrix all of whose entries are zero: $[\mathcal{O}]_{ij} = 0$ for all i, j.

Exs $\begin{bmatrix} 0 & 0 \end{bmatrix}$ $\begin{bmatrix} 0 & 0 & 0 \end{bmatrix}$ $\begin{bmatrix} 0 & 0 \end{bmatrix}$ etc

Ex: $\begin{bmatrix} 0 & 0 \end{bmatrix}$ $\begin{bmatrix} 0 & 0 & 0 & 0 \end{bmatrix}$ $\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$ etc.

<u>Definition</u>: The additive inverse of $A \in M_{mn}$ is the matrix $-A$ with entries

$$[-A]_{ij} = -[A]_{ij}.$$

Ex: $-\begin{bmatrix} 3 & 2 \\ 2 & 1 \end{bmatrix} = \begin{bmatrix} -3 & -2 \\ -2 & -1 \end{bmatrix}.$

Note: $-A = \underbrace{(-1)A}_{\text{scalar multiple of } A \text{ by } \alpha = (-1).}$

Some properties satisfied by these operations.

Let A, B, C be $m \times n$ matrices. Let $\alpha, \beta \in \mathbb{R}$.

1. $A + B = B + A$ commutativity
2. $(A+B)+C = A + (B+C)$ associativity.
3. $A + \mathcal{O} = A$
4. $A + (-A) = \mathcal{O}$.

Check properties 1 & 4 in the next 2 minutes.

5. $\alpha(\beta A) = (\alpha\beta)A$.
6. $\alpha A + \beta A = (\alpha+\beta)A$
7. $\alpha(A+B) = \alpha A + \alpha B$.

8. $1 \cdot A = A$.

Check properties 5 & 7 in the next 2 minutes.

Let's prove property 7, to show how most of these properties can be checked.

Want to show: $\underbrace{\alpha(A+B)}_{m\times n \text{ matrix}} = \underbrace{\alpha A + \alpha B}_{m\times n \text{ matrix}}$

need to show that each entry $[\alpha(A+B)]_{ij}$ equals the corresponding entry $[\alpha A + \alpha B]_{ij}$.

$[\alpha(A+B)]_{ij} = \alpha[A+B]_{ij}$ by def. of scalar mult.

$$= \alpha\left([A]_{ij} + [B]_{ij}\right)$$ by def. of matrix addition.

$$= \alpha[A]_{ij} + \alpha[B]_{ij}$$ by distributivity of mult. of real numbers

$$= [\alpha A]_{ij} + [\alpha B]_{ij}$$ by def of scalar multip.

$$= [\alpha A + \alpha B]_{ij}$$ by def of addition of matrices.

$\square$ ← symbol means our proof is complete.

Notice how we used distributivity of multiplication of real numbers to show distributivity of scalar multiplication of matrices.

multiplication w ...

Definition: Let $A \in M_{m\times n}$. The **transpose** of A, written A^t, is the $n\times m$ matrix with

$$[A^t]_{ij} = [A]_{ji}.$$

(in other words, transposing A swaps its rows & columns)

Ex: $A = \begin{bmatrix} 2 & 2 & 3 \\ 0 & 0 & -1 \end{bmatrix} \in M_{2\times 3}$

$A^t = \Big[\;2 \quad 0 \;\cdots$

$$A = \begin{bmatrix} 2 & 0 \\ 2 & 0 \\ 3 & -1 \end{bmatrix}$$

Question: Prove that $A^t + B^t = (A+B)^t$.

Answer: Need to show that

$$[A^t + B^t]_{ij} = [(A+B)^t]_{ij}$$

for all $1 \le i \le m$
$1 \le j \le n$.

$$[A^t + B^t]_{ij} = [A^t]_{ij} + [B^t]_{ij} \quad \text{(def of matrix add)}$$

$$= [A]_{ji} + [B]_{ji} \quad \text{(def of transpose)}$$

$$= \left[A+B\right]_{ji} \quad \text{(def of matrix add)}$$

$$= \left[(A+B)^t\right]_{ij} \quad \text{(def of transpose)}$$

$\square$.

Theorem: Let $\alpha \in \mathbb{R}$ & let $A \in M_{mn}$

$$(\alpha A)^t = \alpha A^t.$$

Proof: exercise.

Theorem: $(A^t)^t = A$ for any $A \in M_{mn}$.

Proof: $[(A^t)^t]_{ij} = [A^t]_{ji}$ (by def of transpose)

$= [A]_{ij}$ (by def of transpose)

$\square$

Definition: Matrix-vector product.

Let $A \in M_{m \times n}$ with columns $\vec{A}_1 \dots \vec{A}_n$.

Let $\vec{u}$ be a vector of size n. Then

$$\underset{\text{by definition}}{A\vec{u} =} [\vec{u}]_1 \vec{A}_1 + [\vec{u}]_2 \vec{A}_2 + \dots + [\vec{u}]_n \vec{A}_n.$$

Ex: $A = \begin{bmatrix} 3 & 2 & 2 \\ 0 & -1 & 2 \end{bmatrix} \Big\} m=2 \text{ rows}$

$\underbrace{}_{n=3 \text{ columns}}$

$$\vec{A}_1 = \begin{bmatrix} 3 \\ 0 \end{bmatrix} \quad \vec{A}_2 = \begin{bmatrix} 2 \\ -1 \end{bmatrix} \quad \vec{A}_3 = \begin{bmatrix} 2 \\ 2 \end{bmatrix}.$$

$$\vec{u} = \begin{bmatrix} 0 \\ 1 \\ 2 \end{bmatrix}$$

Then $A\vec{u} = [\vec{u}]_1 \vec{A}_1 + [\vec{u}]_2 \vec{A}_2 + [\vec{u}]_3 \vec{A}_3$

$$= 0 \cdot \begin{bmatrix} 3 \\ 0 \end{bmatrix} + 1 \cdot \begin{bmatrix} 2 \\ -1 \end{bmatrix} + 2 \cdot \begin{bmatrix} 2 \\ 2 \end{bmatrix}$$

used scalar mult

$$= \begin{bmatrix} 0 \\ 0 \end{bmatrix} + \begin{bmatrix} 2 \\ -1 \end{bmatrix} + \begin{bmatrix} 4 \\ 4 \end{bmatrix}$$

$$= \begin{bmatrix} 0+2+4 \\ 0-1+4 \end{bmatrix} = \begin{bmatrix} 6 \\ 3 \end{bmatrix}.$$

Notice: If $A \in M_{m \times n}$, then $\vec{u}$ needs to be a vector with $\underline{n}$ entries, and

$A\vec{u}$ will have $\underline{m}$ entries.

<u>Theorem</u> (Systems of linear equations as Matrix Multiplication).

Consider the sys of lin. eq.

$\ldots x_n = b_1$

$$a_{11}x_1 + \dots + a_{1n}x_n$$

$$\vdots$$

$$a_{m1}x_1 + \dots + a_{mn}x_n = b_m.$$

Let $A = \begin{bmatrix} a_{11} & \cdots & a_{1n} \\ a_{m1} & \cdots & a_{mn} \end{bmatrix}$ $\vec{b} = \begin{bmatrix} b_1 \\ \vdots \\ b_m \end{bmatrix}$ $\vec{x} = \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix}$

Then the set of solutions $x_1 \dots x_n$ of our system equals the set of solutions $\vec{x}$ of the matrix equation

$$A \cdot \vec{x} = \vec{b}.$$

(i.e. if $x_1, \dots x_n$ solves our system of linear eq,

then $\vec{x} = \begin{bmatrix} x_1 \\ \vdots \end{bmatrix}$ solves our matrix equation,

$\begin{bmatrix} \vdots \\ x_n \end{bmatrix}$

& vice-versa).

Ex: Solutions to

$$3x_1 + 2x_2 = 0$$
$$5x_1 + 7x_2 = 3$$

are the same as vector solutions to

$$\begin{bmatrix} 3 & 2 \\ 5 & 7 \end{bmatrix} \cdot \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 3 \end{bmatrix}.$$

Definition: Matrix multiplication.

Let $A \in M_{m \times n}$. Let $B \in M_{n \times p}$.

Then $AB \in M_{m \times p}$ is the matrix with entries

$$[AB]_{ij} = \sum_{k=1}^{m} a_{ik} b_{kj}.$$

We can reformulate this definition as follows:

Let B have columns $\vec{B}_1 \ldots \vec{B}_p$. Then

AB is the $m \times p$ matrix with columns

$$A\vec{B}_1, A\vec{B}_2, \ldots, A\vec{B}_p.$$

Ex: $A = \begin{bmatrix} 3 & 2 \\ 1 & 1 \end{bmatrix}$ $\qquad B = \begin{bmatrix} 2 & 5 \\ 2 & 1 \end{bmatrix}$

$\vec{B}_1 = \begin{bmatrix} 2 \\ 2 \end{bmatrix} \quad \vec{B}_2 = \begin{bmatrix} 5 \\ 1 \end{bmatrix}$

$$A\vec{B}_1 = 2\begin{bmatrix}3\\1\end{bmatrix} + 2\begin{bmatrix}2\\1\end{bmatrix} = \begin{bmatrix}2\cdot 3 + 2\cdot 2\\ 2\cdot 1 + 2\cdot 1\end{bmatrix} = \begin{bmatrix}10\\4\end{bmatrix}$$

$$A\vec{B}_2 = 5\begin{bmatrix}3\\1\end{bmatrix} + 1\cdot\begin{bmatrix}2\\1\end{bmatrix} = \begin{bmatrix}5\cdot 3 + 1\cdot 2\\ 5\cdot 1 + 1\cdot 1\end{bmatrix} = \begin{bmatrix}17\\6\end{bmatrix}$$

$$AB = \begin{bmatrix}10 & 17\\ 4 & 6\end{bmatrix}.$$

Note: $AB = \begin{bmatrix}2\cdot 3 + 2\cdot 2 & 5\cdot 3 + 1\cdot 2\\ 2\cdot 1 + 2\cdot 1 & 5\cdot 1 + 1\cdot 1\end{bmatrix}$ as in first version of our definition.

Note: In order that AB be defined, need the number of columns of A

to equal number of rows of B.

i.e. $A \in M_{m \times n}$ $\quad B \in M_{n \times p}$

columns $\quad$ rows

Note: you might expect matrix multiplication to be defined by $[AB]_{ij} = [A]_{ij} \cdot [B]_{ij}$.

This operation turns out to be mostly useless!

Exercise: let $A = \begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix}$ $\quad B = \begin{bmatrix} 3 & 0 \\ 0 & -1 \end{bmatrix}$.

Do we have an equality

$$AB = BA \ ?$$

$$\begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 3 & 0 \\ 0 & -1 \end{bmatrix} = \begin{bmatrix} 2\cdot 3 + 1\cdot 0 & 2\cdot 0 + 1(-1) \\ 1\cdot 3 + 1\cdot 0 & 1\cdot 0 + 1(-1) \end{bmatrix} = \begin{bmatrix} 6 & -1 \\ 3 & -1 \end{bmatrix}$$

$$\begin{bmatrix} 3 & 0 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix} = \begin{bmatrix} 3\cdot 2 + 0\cdot 1 & 3\cdot 1 + 0\cdot 1 \\ 0\cdot 2 - 1(1) & 0\cdot 1 - 1\cdot 1 \end{bmatrix} = \begin{bmatrix} 6 & 3 \\ -1 & -1 \end{bmatrix}$$

hence $AB \neq BA$.